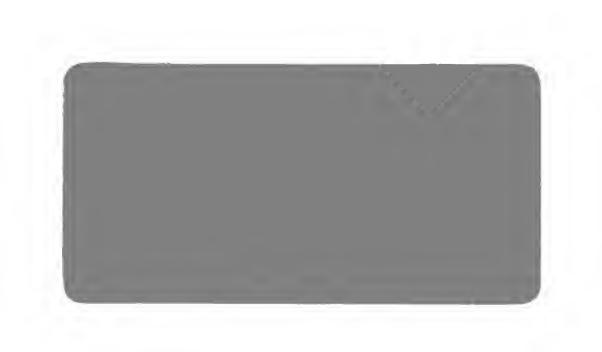# Faculty Working Papers

RANDOM COEFFICIENT, MEASUREMENT ERRORS, AND THE
CAPITAL ASSET PRICING MODEL

Cheng F. Lee, Professor, Department of Finance

#651

College of Commerce and Business Administration
University of Illinois at Urbana-Champaign

RANDOM COEFFICIENT, MEASUREMENT ERRORS, AND THE
CAPITAL ASSET PRICING MODEL

Cheng F. Lee, Professor, Department of Finance

#651

Summary:

Based upon both theoretical and empirical arguments, the Capital Asset
Pricing Model (CAPM) has been specified as a random coefficient and measure-
ment errors rates of return generating process. The impacts of measurement
errors associated with both market rates of return and risk-free rate on the
estimated beta coefficient and estimated random coefficient parameters are
analyzed in detail.

Random Coefficients, Measurement Errors,
and the Capital Asset Pricing Model

Introduction

The main purpose of this paper is to show how a mixed measurement
error and random coefficient model can be used to interpret some empiri-
cal econometric results in a more realistic fashion. The Capital Asset
Pricing Model [CAPM] developed by Sharpe (1964), Lintner (1965) and
Mossin (1966) is used as an example to do the related analyses. It is
shown that the measurement errors of explanatory variable can bias the
estimated slope and the estimated random coefficient parameters.

In the first sections, the theoretical and empirical literature of
the CAPM is reviewed and criticized. In the second section, possible
error components associated with the possible effects of measurement
error on both the market rates of return and risk-free rates of interest
are defined. In the third section, the reason of using a random coefficient
instead of a fixed coefficient CAPM is justified in accordance with
the results of section II. The measurement errors of both market
rates of return and risk-free rates are then introduced to the random
coefficient CAPM. The impacts of these measurement errors on the
estimated random beta coefficient is then analyzed in detail. In the
fourth section, the effects of measurement errors associated with
excess market rates of return on the estimated random coefficient param-
eters are analyzed in accordance with the technique developed by Griliches
and Ringstad (1970). Possible implications of the results developed in
this section on Fama and Macbeth's (1973) and Treynor and Mazny's (1966)

empirical results are also discussed. Finally, results of this study
are summarized and concluding remarks are indicated.

I. Review and Critique of the CAPM

There exist several approaches to the determination of capital asset
prices under conditions of uncertainty. The mean-variance model is due to
Markowitz (1952), Sharpe (1964), Linterner (1965) and Mossin (1966); the
state preference model is due to Arrow (1964) and Debreu (1959). Most
recently, Ross (1976, 1977) has used the arbitrage theory to derive the
CAPM. Ross has argued that the arbitrage approach can be regarded as a
compromise between the mean-variance approach and the state preference
approach. Although the mean-variance model is less elegant than the
state preference model and the arbitrage model, it does allow us to do
empirical work.

Specifying an individual utility function which contains the mean and
the standard deviation of the returns of assets and investment opportunity
set, Sharpe (1964) derived the CAPM. In deriving the CAPM, Sharpe has
also assumed the existence of the risk-free rate of interest. The CAPM
is defined as

$$R_j = R_F + \beta_j (R_M - R_F) , \tag{1}$$

where R_j = expected rate of return on the jth assets,

$\quad R_F$ = risk free rate of interest,

$\quad R_M$ = rate of return on a "market portfolio" consisting of an invest-
ment in every asset outstanding in proportion to its value,

$$\beta_j = cov(R_j, R_m)/\sigma^2(R_m) = \text{the systematic risk of the } j^{th} \text{ asset.}$$

Lintner (1965) derived the CAPM in the following form

$$R_j - R_F = \frac{(R_m - R_F)}{\sigma^2(R_m)} \sum_{i=1}^{n} x_i cov(R_j, R_i)$$

$$= (R_m - R_F)[\beta_j + \frac{x_i \sigma^2(\varepsilon_i)}{\sigma^2(R_m)}] , \qquad (2)$$

where ε_i = the disturbance term of Sharpe's market model.

Fama (1968) has shown that the difference between (1) and (2) are trivial. Mossin (1966), Jensen (1972), and others have derived the economic implications of the single period CAPM from the general equalibrium framework; they are

(i)   For n-1 risky assets we have n-1 CAPM's; in addition, we also have a budget equation. These n equations formulate a complete demand system.

(ii)   In equilibrium, prices must be such that each individual will hold the same percentage of the total outstanding stock of all risky assets; therefore, CAPM can be identified without the information of the functional form of utility function.

(iii)   The CAPM is independent of the initial holdings in the individual's asset.

(iv)   The CAPM is free from aggregation problems.

Although the CAPM is an elegant method used in investigating portfolio performance, valuation theory, determination of the "cost of capital" and

corporate investment decisions etc., it still faces many theoretical and empirical problems.

Theoretically, the CAPM is based essentially upon the mean-standard deviation utility analysis. In accordance with Tobin (1969) and Tsiang (1972) and Levy and Markowitz (1979), this analysis is justified if and only if either (i) the investor's utility function is quadratic or (ii) the investor regards the uncertain outcomes as all normally distributed or (iii) the aggregate risk taken by the individual concerned is small compared with his total wealth, including his physical, financial as well as human wealth.

Empirically, Jensen (1968, 1969) has proved that (1) can be written as

$$R_{jt} = R_{Ft}(1 - \beta_j) + \beta_j R_{Mt} + \epsilon_{jt}, \tag{3}$$

where ϵ_{jt} is a disturbance term. The subscript t indicates the time series observations, of each variable. (3) implies that a single period model will be employed to a set of multiperiod data. Roll (1969) has argued that the main problem in applying the one period model to time series data is the neglect of the change of wealth over time. This situation will make the systematic risk either stationary in a stochastic sense or non-stationary. Empirically, CAPM also faces errors-in-variables, specification, the time horizon and the random coefficient problems. In accordance with the constrained regression theory, Jensen defined a regression model as

$$R_{jt} - R_{Ft} = \alpha_j + \beta_j(R_{mt} - R_{Ft}) + \epsilon_{jt}, \tag{4}$$

where α_j is the Jensen performance measure and β_j is the systematic risk. In addition to the Jensen performance measure, two other performance measures are

(a) $\quad \dfrac{\alpha_j}{\beta_j} = \dfrac{E(R_j) - R_F}{\beta_j} - [E(R_m) - R_F]$

(5)

(b) $\quad \dfrac{E(R_j) - R_F}{\sigma(R_j)} = \dfrac{\alpha_j}{\sigma(R_j)} + \dfrac{E(R_m) - R_F}{\sigma(R_m)}$,

where $\dfrac{\alpha_j}{\beta_j}$ is called Treynor's (1961) measure and $\dfrac{E(R_j) - R_F}{\sigma(R_j)}$ is called Sharpe's (1966) measure.

Levy (1972) has shown that basic time unit being different with the true investment time horizon will bias the estimate of investment performance. He concluded that reward-to-variability [Sharpe Performance Measure] will be downward biased for a portfolio with higher risk and upward biased for a portfolio with lower risk when the basic time unit is larger than the true time horizon. Levhari and Levy (1977) has shown that the Treynor measure has subjected to same investment horizon bias as the Sharpe performance measure does.

Referring to the specification analysis, there exist three sources of bias, i.e. (i) the single period bias; this bias is due to the employment of multi-period data in a single period model; (ii) the mean-standard deviation utility bias; this bias is due to the fact that the utility function is assumed to include only two arguments, the mean and the standard deviation of the rate of return and (iii) one factor model bias; this bias assets that the one factor specification is too simple to explain the real world. The one factor model has been extended to two-factor, three-factor and four-factor models. These models will be described and criticized as follows:

(A) The two-factor model, which is due to Black (1972) and Black, Jensen, and Scholes (1973) can be defined as

$$R_{jt} = R_{zt}(1 - \beta_j) + \beta_j R_{mt} + \varepsilon_{jt},$$

(6)

where R_{zt} is called zero beta factor, it is defined as the return on a portfolio which has a zero covariance with R_{mt}.

(B) The four-factor model, which is due to Fama and Macbeth (1973) can be written as

$$R_{jt} = \gamma_{0t} + \gamma_{1t}\beta_j + \gamma_{2t}\overline{\beta}_j^2 + \tilde{\gamma}_{3t}\overline{\sigma}_j(U) , \qquad (7)$$

where γ_{0t} plays the same role as R_{zt},

$\overline{\beta}_j^2$ is the average of the β_j^2 for all individual security in portfolio j.

$\overline{\sigma}_j(U)$ is the average of the residual standard deviations from the market model for all securities in portfolio j.

(C) The three-factor model, which was developed by Merton (1973) can be defined

$$E(R_j) = R_F + \lambda_1[E(R_M) - R_F] + \lambda_2[E(R_N) - R_F] , \qquad (8)$$

where R_N = the return on the asset which is negatively correlated with changes in the riskless interest rate.

ρ_{NM} = the correlation coefficient between R_N and R_M.

$$\beta_{jM} = \frac{cov(R_j, R_M)}{\sigma_M^2}$$

$$\beta_{jN} = \frac{cov(R_j, R_N)}{\sigma_N^2}$$

$$\beta_{NM} = \frac{cov(R_N, R_M)}{\sigma_M^2} , \qquad \beta_{MN} = \frac{cov(R_N, R_M)}{\sigma_N^2}$$

$$\lambda_1 = \frac{(\beta_{jM} - \beta_{jN}\beta_{NM})}{1 - \rho_{NM}^2}$$

$$\lambda_2 = \frac{(\beta_{jN} - \beta_{jM}\beta_{MN})}{1 - \rho_{NM}^2}.$$

The three-factor model is derived in the context of the continuous time intertemporal asset pricing model, this model allows the shifting of the investment opportunity set. Merton assumed that the shift of the investment opportunity set can be characterized by changes of the riskless rate of interest.

Equation (8) is a theoretical model, it can be shown that λ_1 and λ_2 are multiple regression coefficients as defined in equation (9).

$$R_{jt} = \alpha_0 + \alpha_1 R_{Mt} + \alpha_2 R_{Nt} + \varepsilon_{jt} \tag{9}$$

Based upon the specification analysis, we have

(A) $\beta_{jM} = \alpha_1 + \alpha_2 \beta_{NM}$

$$\tag{10}$$

(B) $\beta_{jN} = \alpha_2 + \alpha_1 \beta_{MN}$

From equations (10A) and (10B), we have

$$\alpha_1 = \frac{\beta_{jM} - \beta_{jN}\beta_{NM}}{1 - (\beta_{NM})(\beta_{MN})} = \frac{\beta_{jM} - \beta_{jN}\beta_{NM}}{1 - \rho_{NM}^2} = \lambda_1$$

$$\alpha_2 = \frac{\beta_{jN} - \beta_{jM}\beta_{NM}}{1 - (\beta_{NM})(\beta_{MN})} = \frac{\beta_{jN} - \beta_{jM}\beta_{MN}}{1 - \rho_{NM}^2} = \lambda_2$$

Therefore, Merton's (1973) three-factor model is empirically identical to Stone's (1974) two-index model.

The three-factor model has been indirectly tested by Lloyd and Shick (1977) and Lynge and Zumwalt (1979). Empirically, Fama and Macbeth (1973)

have shown that four-factor model is not significantly different from two-factor model; Blume and Friend (1973) has shown that Black's (1972) two-factor model does not perform better than Sharpe (1964) and Lintner's (1965) one factor model.

Although the measurement error problem of the CAPM has been investigated by Roll (1969), Friend and Blume (1970), Miller and Scholes (1972) and Lee and Jen (1978) etc., systematic research based upon the errors-in-variables model and random coefficient model still remain to be done.

In order to begin such an investigation, I will justify the empirical model to be employed in this paper as follows:

There exists three different approaches to estimate systematic risk from (3), i.e., (a) Regressing R_{jt} on R_{mt} without any restriction, (b) Regressing $(R_{jt} - R_{Ft})$ on $(R_{mt} - R_{Ft})$, explicitly employing the linear constraint $(1 - \beta) + \beta = 1$, and (c) Regressing R_{jt} on R_{Mt} by assuming R_{Ft} is constant over time.

For econometric theory, it can be easily shown that approach (a) will lose efficiency and obtain an estimator different from the constraint estimator, approach (c) will induce specification error, as shown by Roll (1969), and that the non-systematic risk which was obtained from both (a) and (c) will be different from that of (b). Therefore, we will use approach (b) as the basic specification in this paper.

II. Possible Error Components of Market Rate of Return and Risk Free Rate of Interest.

One of the basic assumptions used to derive CAPM is that there exists no transaction cost and no taxes. Clearly, these are the essential components of measurement error in a narrow sense. This kind of error applies to R_j, R_m, and R_F. In addition, in the real world, the

true R_m and R_F cannot be observed. In general, the New York Stock Exchange (NYSE) average is used as a proxy of R_m and the monthly 90 days treasury bill rate as a proxy of R_F. The error caused by the proxy variables is called the measurement error in the wide sense. A potential source of measurement error on R_m is that the NYSE Average only includes a subset of the market portfolio. The possible source of measurement error on R_F are that treasury bill rate is risk-free only in the default sense, and in the real world, investors can not borrow and lend an unlimited amount at an exogenously given risk-free rate of interest R_F. Additional discussion of the measurement errors of R_m and R_F are discussed by Lee and Jen (1978), Brennan (1971) and Roll (1969) etc.

For simplicity, we will rewrite (4) as

$$V_{jt} = \alpha_j + \beta_j U_{mt} + \epsilon_{jt} \tag{11}$$

where $V_{jt} = R_{jt} - R_{bt}$

$U_{mt} = R_{mt} - R_{bt}$

R_{bt} = a weighted average of market's borrowing and lending rate as defined by Brennen (1971).

Both V_{jt} and U_{jt} are unobserved, we can only observe Y_{jt} and X_{jt}. The relationship between the true values and the observed values can be defined as

$$Y_{jt} = V_{jt} + \tau_{jt} \, ,$$

$$\tag{12}$$

$$X_{mt} = U_{mt} + \eta_{mt} \, ,$$

where $\tau_{jt} = R_{bt} - R_{Tt} = E + \tau_t, \quad \eta_{mt} = \tau_{jt} + R_{mt} - Y_{mt} \, ,$

$$R_{mt} - \gamma_{mt} = \Psi - \lambda_t, \quad \tau_{jt} \sim N(E, \sigma_\tau^2), \quad cov(\tau_{jt}, \eta_{mt}) = \sigma_\tau^2 ,$$

$$\eta_{mt} \sim N(E - \Psi, \sigma_\eta^2), \quad \sigma_\eta^2 = \sigma_\tau^2 + \sigma_\lambda^2,$$

E and Ψ are constant measurement error of R_{Tt} and α_{mt} respectively.

R_{Tt} = monthly treasury bill rate,

γ_{mt} = the market rate of return calculated from NYSE average

According to Jen and Lee (1978), E is always greater than zero, the sign of Ψ is ambiguous. τ_{jt} and η_{mt} are random measurement errors of V_{jt} and U_{mt} respectively. The errors-in-variables model of this kind is not entirely consistent to the classical case discussed by Johnston (1972) and others.

III. Effects of Measurement Error on Systematic Risk in a Random Coefficient Model

Relaxing the restrictive assumption, we will allow systematic β_j to be stochastic.

The best reason of using random instead of fixed coefficient model has been explored by Hildreth and Honck (1968). They have argued that the random coefficient assumption can essentially be used to take care of the variation of the coefficient associated with the omitted variable. In the second section, the so-called three-factor or four-factor model has been reviewed in some detail. In addition, Sharpe (1977) and Ross (1976, 1977) have developed more general multi-factor models. If a single-factor instead of a multi-factor CAPM is used, then it is reasonable to allow the beta coefficient β_j to become a random parameter. Now, the interpretation random β_j coefficient is discussed.

If excess market return increases by one unit, all other factors remain constant, excess asset return may respond randomly increase with a certain

mean and a positive variance. Thus (12) can be rewritten as

$$V_{jt} = b_{jt}U_{mt} + \varepsilon_{jt} \tag{13}$$

$$E\begin{pmatrix} \varepsilon_t \\ b_{jt} \end{pmatrix} = \begin{pmatrix} 0 \\ \beta_j \end{pmatrix}, \qquad V\begin{pmatrix} \varepsilon_t \\ b_{jt} \end{pmatrix} = \begin{pmatrix} \sigma_0^2 & 0 \\ 0 & \sigma_1^2 \end{pmatrix} \qquad t=1,\ldots,n$$

$$E\left(\begin{pmatrix} \varepsilon_t \\ b_{jt}-\beta_j \end{pmatrix} [\varepsilon_i, b_{jt}-\beta_j]\right) = 0 \qquad\qquad \text{for } t \neq i$$

Let $v_{jt} = V_{jt} - \overline{V}$, $u_{mt} = U_{mt} - \overline{U}$, and substitute them into (10), we have

$$v_{jt} = \beta_j u_{mt} + \varepsilon_{jt}^* \tag{14}$$

where $\varepsilon_{jt}^* = \varepsilon_{jt} + (b_{jt} - \beta_j)u_{mt}$.

Following Theil and Mennes (1959), the most efficient estimator of b_{jt} is

$$b_{jt} = \frac{\displaystyle\sum_{t=1}^{n} \frac{u_t v_t}{\sigma_0^2 + \sigma_1^2 v_t^2}}{\displaystyle\sum_{t=1}^{n} \frac{u_t^2}{\sigma_0^2 + \sigma_1^2 v_t^2}} \tag{15}$$

Since u_t and v_t are unobserved, we should substitute for them by $x_t = X_t - \overline{X}$ and $y_t = Y_t - \overline{Y}$ respectively. Therefore, the observed b_{jt} can be estimated by

$$b_{jt}' = \frac{\displaystyle\sum_{t=1}^{n} \frac{x_t y_t}{\sigma_0^2 + \sigma_1^2 x_t^2}}{\displaystyle\sum_{t=1}^{n} \frac{x_t^2}{\sigma_0^2 + \sigma_1^2 x_t^2}} . \tag{16}$$

When both σ_0^2 and σ_1^2 are known the relationship between plim b_{jt} and plim b_{jt}' can be derived as follows:

(i)   Substituting (14) into (15) and taking the probability limit of b_{jt}, then we have

$$\text{plim } b_{jt} = \frac{\text{plim}\left(\beta_j \sum_{t=1}^{n} \frac{u_t^2}{\sigma_0^2+\sigma_1^2 u_t^2} + \sum_{t=1}^{n} \frac{\varepsilon_{jt}^* u_t}{\sigma_0^2+\sigma_1^2 u_t^2}\right)}{\text{plim}\left(\sum_{t=1}^{n} \frac{u_t^2}{\sigma_0^2+\sigma_1^2 u_t^2}\right)} \tag{17}$$

if   plim $\left(\sum_{t=1}^{n} \dfrac{u_t^2}{\sigma_0^2+\sigma_1^2 u_t^2}\right)$ exists,

then plim $b_{jt} = \beta_j$.

Equation (17) implies that the OLS estimates of systematic risk is a consistent estimate if u is free from the measurement errors.

(ii)   Substituting (12) into (16) and taking the probability limit of b_{jt}', then we have

$$\text{plim } b_{jt}' = \frac{\beta_j + \dfrac{\text{plim} \sum_{t=1}^{n} \dfrac{\tau_t'^2}{\sigma_0^2+\sigma_1^2 x_t^2}}{\text{plim} \sum_{t=1}^{n} \dfrac{u_t^2}{\sigma_0^2+\sigma_1^2 x_t^2}}}{1 + \dfrac{\text{plim} \sum_{t=1}^{n} \dfrac{\eta_t'^2}{\sigma_0^2+\sigma_1^2 x_t^2}}{\text{plim} \sum_{t=1}^{n} \dfrac{u_t^2}{\sigma_0^2+\sigma_1^2 x_t^2}}} \tag{18}$$

where $\tau_t' = \tau_t - E$, $\eta_t' = \eta_t - E + \psi = \tau_t' + \lambda_t$.

Equations (17) and (18) can be used to estimate the bias of estimated systematic risk associated with the random-coefficient-errors-in-variables model as [1].

$$\text{plim } b_{jt}' - \text{plim } b_{jt} = \frac{D(1-\beta_j) - \beta_j C}{1 + C + D} \qquad (19)$$

where

$$C = \frac{\text{plim} \sum_{t=1}^{n} \dfrac{\lambda_t^2}{\sigma_0^2 + \sigma_1^2 x_t^2}}{\text{plim} \sum_{t=1}^{n} \dfrac{u_t^2}{\sigma_0^2 + \sigma_1^2 x_t^2}}$$

$$D = \frac{\text{plim} \sum_{t=1}^{n} \dfrac{\tau_t^2}{\sigma_0^2 + \sigma_1^2 x_t^2}}{\text{plim} \sum_{t=1}^{n} \dfrac{u_t^2}{\sigma_0^2 + \sigma_1^2 x_t^2}} \; .$$

If $\sigma_1^2 = 0$, then (19) reduces to the fixed coefficient errors-in-variables case, i.e. [2]

$$\text{plim } b_{jt}' - \text{plim } b_{jt} = \frac{(1-\beta_j) \dfrac{\sigma_{\tau't}^2}{\sigma_{u_t}^2} - \beta_j \dfrac{\sigma_{\lambda t}^2}{\sigma_{u_t}^2}}{1 + \dfrac{\sigma_{\tau't}^2 + \sigma_{\lambda t}^2}{\sigma_{u_t}^2}}$$

$$= \frac{(1-\beta_j)\sigma_{\tau' t}^2 - \beta_j \sigma_{\lambda_t}^2}{\sigma_{u_t}^2 + \sigma_{\tau' t}^2 + \sigma_{\lambda_t}^2} \quad . \tag{20}$$

If $\sigma_0^2 = 0$, then (19) reduces to

$$\text{plim } b_{jt}' - \text{plim } b_{jt} = \frac{D'(1-\beta_j) - \beta_j C'}{1 + C' + D'} \tag{21}$$

where

$$C' = \frac{\text{plim} \sum_{t=1}^{n} \frac{\lambda_t^2}{x_t^2}}{\text{plim} \sum_{t=1}^{n} \frac{u_t^2}{x_t^2}}$$

$$D' = \frac{\text{plim} \sum_{t=1}^{n} \frac{\tau_t'^2}{x_t^2}}{\text{plim} \sum_{t=1}^{n} \frac{u_t^2}{x_t^2}} \quad .$$

Under above mentioned two circumstances, it also should be noted that equation (16) reduces to

$$b_{jt}' = \frac{\sum_{t=1}^{n} \frac{y_t}{x_t}}{n} \quad . \tag{22}$$

Essentially, this is a combined ratio estimator. In other words, for $\sigma_0^2 = 0$ or $\sigma_1^2 = 0$, the knowledge of σ_0^2 and σ_1^2 is not required at all.

When both σ_0^2 and σ_1^2 are not known, following Theil and Mennes (1959), then they can be estimated by using the residuals obtained from ordinary least squares and x_t. Under these circumstance, both $\hat{\sigma}_0^2$ and $\hat{\sigma}_1^2$ are affected by the measurement errors of y_t and x_t. The effects of measurement errors in both x_t and y_t on $\hat{\sigma}_0^2$ and $\hat{\sigma}_1^2$ will be analyzed in the following section.

IV. Effects of Measurement Errors on the Estimated Random Coefficient Estimators

To the best knowledge of this author, Lee's (1973) dissertation was the first study to develop a random-coefficient-errors-in-variables [RCEV] CAPM as discussed in the previous section. Fabozzi and Francis (1978) and Sunder (1980) have used the random coefficient model developed by Theil and Mennes (1959) to investigate random nature of beta coefficient in terms of Market Model. However, they have entirely neglected the possible impacts of measurement errors associated with R_m and R_f on the estimated beta coefficients. Roll (1969, 1977) and Lee and Jen (1978) have argued that both R_m and R_f are measured with errors and therefore, X_{mt} as indicated in equation (12) are measured with errors.

Following Theil (1971) and Francis and Fabozzi (1978), the model used to estimate the parameter σ_0^2 and σ_1^2 as indicated in equation (13) can be defined as

$$\hat{\epsilon}_{jt}^{*2} = \sigma_0^2 P_t + \sigma_1^2 Q_t + f_{jt} \qquad (23)$$

where $\hat{\epsilon}_{jt}^{*}$ is the estimated ordinary least square (OLS) residual from equation (14); f_{jt} is the residual term for the multiple regression of equation (23). Finally, P_t and Q_t are defined as

(A) $P_t = 1 - u_t/\Sigma u_t{}^2$

$$(24)$$

(B) $Q_t = u_t{}^2 \cdot [1 - 2(u_t{}^2/(\Sigma u_t)^2 + u_t{}^4(\Sigma u_t{}^2)^2]$

If the sample size is large enough, then equation (24) can be approximately defined as   [See Theil and Mennes (1959) for detail]

$$\hat{\varepsilon}^{*2}_{jt} = \sigma_0^2 + \sigma_1^2 u_t^2 + f_{jt} \qquad (25)$$

However, both V_{jt} and U_{mt} are not observable and y_t and x_t are used to replace them to estimate $\hat{\varepsilon}^{*2}_{jt}$, σ_0^2 and σ_1^2. Therefore, the RCEV CAPM can be defined as

$$y_{jt} = \beta_j x_t + [-\beta_j \eta_{mt} + (b_{jt} - \beta_j)x_t + \varepsilon_{jt}] \qquad (26)$$

Under this circumstance, it is easy to show that equation (25) in terms of y_t and x_t can be rewritten as[3]

$$\hat{\delta}^{'2}_{jt} = (\sigma_0^2 + \beta_j^2 \sigma_\eta^2) + \sigma_1^2 x_t^2 + [-\sigma_1^2 \eta^2 - 2\sigma_1^2 u\eta + f_{jt}] \qquad (27)$$

where $\hat{\delta}_{jt}$ is the estimated OLS residuals associated with equation (26).

Equation (27) implies that the expected value of $\hat{\delta}_{jt}^2$ can be defined as

$$E(\hat{\delta}_{jt}^2) = \sigma_0^2 + \beta_j^2 \sigma_\eta^2 + \sigma_1^2 \{E(x_t^2) - \eta^2\}$$

where σ_0^2 is the pure OLS residual variance; $\beta_j^2 \sigma_\eta^2$ is the variance associated with the measurement errors of excess market rates of return; $\sigma_1^2 \{E(x_t^2) - \eta^2\}$ is the variance associated with the random coefficient systematic risk. It is clear that σ_0^2 is the nonsystematic risk. However, $\beta_j^2 \sigma_\eta^2$ and $\sigma_1^2 \{E(x_t^2) - \eta^2\}$ are not necessary nonsystematic risks.

Therefore, the standard OLS regression two-component risk decomposition technique can no longer apply to the RCEV CAPM model.

The impact of the measurement errors of x_t on the estimated σ_1^2 is now analyzed. Following Griliches and Ringstad (1970), the variables, u_t, x_t, and η are parameterized as follows.[3]

Since $\bar{x}_t = 0$ and hence $\mu = \bar{u}_t = 0$, and that we parameterize our problem in such that $\sigma_x^2 = 1$, $\sigma_\eta^2 = \lambda < 1$, and hence $\sigma_u^2 = 1 - \lambda < 1$.
Thus

$$u \sim N(0, 1 - \lambda),$$

$$\eta \sim N(0, \lambda)$$

$$x \sim N(0, 1)$$

and therefore

$$x^2 \sim \chi^2(1, 2)$$

$$\eta^2 \sim \chi^2(\lambda, 2\lambda^2)$$

From Theil (1957, 1971), it can be shown that

$$E(\hat{\sigma}_1^2 - \sigma_1^2) = -\sigma_1^2 b_{\eta^2 x^2} - 2\sigma_1^2 b_{(u\eta)x^2} = \text{plim}(\hat{\sigma}_1^2 - \sigma_1) \tag{28}$$

where $b_{\eta^2 x^2}$ and $b_{(u\eta)(x^2)}$ are auxiliary regression coefficients. Moreover, given the above assumptions and definitions, we also have

$$\text{cov}(u\eta)x^2 = E(u\eta)(u^2 + 2u\eta + \eta^2) - (Eu\eta)(Ex^2)$$

$$= 2Eu^2\eta^2 = 2(Eu^2\eta^2) = 2\lambda(1 - \lambda) \tag{29}$$

and

$$\text{cov}(\eta^2 x^2) = E(\eta^2)(u^2 + 2u\eta + \eta^2) - (E\eta^2)[E(u^2 + 2u\eta + \eta^2)]$$

$$= E(\eta^4) - (E\eta^2)^2 = \text{Var}(\eta^2) = 2\lambda^2 \tag{30}$$

Therefore

$$b\eta^2 x^2 = \frac{\text{Var}(\eta^2)}{2} = \lambda^2 \tag{31}$$

$$b(u\eta)(x^2) = \frac{\text{Cov}(u\eta)x^2}{\text{Var}x^2} = \lambda(1 - \lambda) \tag{32}$$

Substituting equations (31) and (32) into equation (28), we obtain

$$\text{plim}(\hat{\sigma}_1^2 - \sigma_1^2) = -2\sigma_1^2\lambda(1 - \lambda) - \sigma_1^2\lambda^2$$

$$= -\sigma_1^2\lambda(2 - \lambda) \tag{33}$$

Equation (33) implies that

$$\hat{\sigma}_1^2 \sim \sigma_1^2(1 - 2\lambda + \lambda^2) = \sigma_1^2(1 - \lambda)^2 \tag{34}$$

Equation (34) implies that the bias associated with $\hat{\sigma}_1^2$ is $(1 - \lambda)^2$ where λ is the fraction of error variance in the total variance in the observed variable. Thus, the problem of errors-in-variables is significantly more serious for the non-linear term since the bias associated with a linear term is only $(1 - \lambda)$ [See Griliches and Ringstad (1970)]. If $\lambda = \cdot 2$, then $(\hat{\sigma}_1^2/\sigma_1^2) \sim \cdot 64$. This result implies that Fabozzi and Francis's (1978) and Saunder's (1980) estimates of $\hat{\sigma}_1^2$ are potentially downward biased. In addition, Cochran (1968) has shown that the measurement errors of regressors can reduce the coefficient of determination $(\overline{R}^2)$ for a regression, this argument has also implied that the t values and $\overline{R}^2$'s associated with Francis and Fabozzi's (1968) and Saunder's (1980) empirical results are also downward biased.

The results obtained in this section can also be used to comment on other empirical researchs. First, Fama and Macbeth (1973) have used the so-called four-factor model as indicated in equation (7) to test the efficiency of capital market. If the estimated systematic risk for individual security is normally distributed, then the squared of the estimated systematic risk will follow a χ^2 distribution. Therefore, $\overline{\beta}_j^2$ also follows a χ^2 distribution. The estimated β_j and $\overline{\beta}_j^2$ are generally measured with errors, and therefore, the estimated γ_{2t} is a downward biased estimator. Secondly, Treynor and Mazuy (1965) have used a square term to test whether mutual fund's rates of return process is linear or not. If the market rates of return used by Treynor and Mazuy are measured with errors, then the estimated coefficient associated with their empirical results are generally downward biased.

V. Summary and Concluding Remarks

A most generalized model for capital asset pricing called RCEV CAPM is developed in accordance with the capital market theory, the specification analysis, and the nature of the data associated with the excess market rates of return. The possible impacts of the measurement errors associated with the excess market rates of return on the random coefficient parameter are analyzed in accordance with the χ^2 distribution. The results are also used to analyze the efficient market hypothesis test done by Fama and Macbeth (1973) and the empirical mutual fund rates of return generating process investigated by Treynor and Mazuy's (1966).

In sum, it has demonstrated that errors-in-variables problem is important in performing the empirical study in finance research; it has

also shown that the random coefficient model is generally more sensitive to the problem of errors-in-variables than the fixed coefficient model does. The trade-off between the random coefficient model and the fixed coefficient model in terms of the errors-in-variables model will be explored in the future research.

M/E/196

Footnotes

[1]Equation (19) implies that the OLS estimator is no longer a consistent estimator unless

$$D(1 - \beta_j) - \beta_j C = 0 \qquad\qquad (A)$$

Equation (A) implies that

$$\beta_j/1 - \beta_j = \frac{C}{D} = \frac{\text{plim } \Sigma \; \lambda_t^2/\sigma_0^2 + \sigma_1^2 x_t^2}{\text{plim } \Sigma \; \tau_t^2/\sigma_0^2 + \sigma_1^2 x_t^2} \qquad\qquad (B)$$

[2]The relative magnitude between equation (19) and equation (20) can be used to determine whether the random coefficient or the fixed coefficient model should be used to estimate the systematic risk of CAPM. The derivation of this criteria will be done in the future research.

[3]Following the assumption of Griliches and Ringstad (1970), it is assumed that u_t, x_t, and η are all normally distributed with zero means and variances σ_u^2, σ_x^2, and σ_η^2.

References

1. Arrow, K. J. (1964), "The Role of Securities in the Optimal of Risk-Bearing, Review of Economic Studies, 31, 91-96.

2. Black, F. (1972), "Capital Market Equilibrium With Restricted Borrowing," Journal of Business, 45, 444-452.

3. Brennan, M. J. (1971), "Capital Market Equilibrium with Divergent Borrowing and Lending Rate," Journal of Financial and Quantitative Analysis, 7, 1197-1205.

4. Cochran, W. G. (1970), "Some Effects of Errors of Measurement on Multiple Correlation," Journal of American Statistical Association, 65, 22-37.

5. Debrew, G. (1959), Theory of Value, New York: John Wiley and Sons Inc., 1959, Chapter 7.

6. Fama, E. F. (1968), "Risk, Return and Equilibrium: Some Clarifying Comments," Journal of Finance, 23, 29, 40.

7. Fama, E. F. and J. D. Macbeth (1973) "Risk, Return, and Equilibrium: Empirical Tests," Journal of Political Economy, 71, 607-636.

8. Friend, I. and Blume, M. (1970), "Measurement of Portfolio Performance Under Uncertainty," The American Economic Review, 60, 561-75.

9. Fabozzi, F. J. and J. C. Francis (1978), "Beta as Random Coefficient," Journal of Financial and Quantitative Analysis, 13, 101-114.

10. Griliches, Z. and V. Ringstad (1970), "Errors-in-variables Bias in Non-linear Context," Econometrica, 38, 368-370.

11. Hildreth, C. and J. P. Houck (1968), "Some Estimators for a Linear Model with Random Coefficients," Journal of the American Statistical Association, 63, 584-595.

12. Jensen, M. (1968), "The Performance of Mutual Funds in the Period 1954-1964," Journal of Finance, 23, 389-416.

13. Jensen, M. (1969), "Risk, the Pricing of Capital Assets, and the Evaluation of Investment Portfolios," Journal of Business, 42, 167-247.

14. Jensen, M. C. (1972), "Capital Markets," The Bell Journal of Economics and Management Science, 3, 357-398.

15. Johnston, J. (1972), Econometric Methods (2nd. ed.), New York: McGraw-Hill.

16. Lee, Cheng F. (1973), "Errors-in-variables Estimation Procedures with Applicators to a Capital Asset Pricing Model," Dissertation submitted to Graduate School, State University of New York at Buffalo, 1973.

17. Lee, Cheng F. and F. Jen (1978), "Effects of Measurement Errors on Systematic Risk and Performance Measure of a Portfolio," Journal of Financial and Quantitative Analysis, 13, 299-312.

18. Levhari, D. and H. Levy (1977), "The Capital Asset Pricing Model and the Investment Horizon," The Review of Economics and Statistics, 49, 92-104.

19. Levy, H. (1972), "Portfolio Performance and The Investment Horizon," Management Science, 18, 645-653.

20. Levy, H. and H. M. Markowitz (1979), "Approximating Expected Utility Function by A Function of Mean and Variance," American Economic Review, 69, 308-317.

21. Lloyd, W. P. and R. A. Shick (1977), "A Test of Stone's Two-Index Models of Returns" Journal of Financial and Quantitative Analysis, 12, 363-76.

22. Lynge, M. J. and J. K. Zumwalt (1979), "An Empirical Study of the Interest Rate Sensitivity of Commercial Bank Return: A Multi-Index Approach," Journal of Financial and Quantitative Analysis, forthcoming.

23. Lintner, J. (1965), "Security Prices, Risk and Maximal Gains from Diversification," Journal of Finance, 20, pp. 587-616.

24. Markowitz, H. M. (1952), "Portfolio Selection," Journal of Finance, 7, 77-91.

25. Merton, R. C. (1973), "Intertemporal Capital Asset Pricing Model," Econometrica, 41, 867-887.

26. Miller, M. H. and Scholes, M. (1972), "Rates of Return in Relation to Risk: A Reexamination of Some Recent Findings," in M. C. Jensen, ed., Studies in the Theory of Capital Market, New York: Praeger Publishers, (1972).

27. Mossin, J. (1966), "Equilibrium in a Capital Asset Market," Econometrica, 34, 768-783.

28. Roll, Richard (1977), "A Critique of the Asset Pricing Theory's Tests," Journal of Financial Economics, 4, p. 129-176.

29. Roll, R. (1969), "Bias in Fitting the Sharpe Model to Time Series Data," Journal of Financial and Quantitative Analysis, 4, 274-289.

30. Ross, S. A. (1976), "Arbitrage Theory of Capital Asset Pricing," Journal of Economic Theory, 8, 341-360.

31. Ross, S. A. (1977), "Return, Risk and Arbitrage," in I. Friend and J. L. Bicksler, Risk and Return in Finance, Vol. 1 (edited), Ballinger Publishing Company.

32. Sharpe, W. F. (1964), "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risks," Journal of Finance, 19, 425-442.

33. Sharpe, W. F. (1966), "Mutual Fund Performance," Journal of Business, (January, 1966), 119-138.

34. Sharpe, W. (1977), "The Capital Asset Pricing Model: A Multi-Beta Interpretation," in Levy, H. and M. Sarnat's Readings book from Academic Press.

35. Stone, B. (1974), "Systematic Interest-Rate Risk in a Two-Index Model of Return," Journal of Financial and Quantitative Analysis, 10, 709-721.

36. Sunder, S. (1980), "Stationarity of Market Risk: Random Coefficients Tests for Individual Stocks," Journal of Finance, Forthcoming.

37. Theil, H. (1971), Principles of Econometrics, New York: John Wiley and Sons, Inc.

38. Theil, H. and L. B. M. Mennes (1959), "Conception Stochastique de Coefficients Multiplicateurs dans l'adjustement Linéarire des Séries Temporelles," Publications del l'Institut de Statistique de l'Universite de Paris, 8, pp. 221-227.

39. Treynor, J. L. (1965), "How to Rate Management of Investment Funds," Harvard Business Review (January-February, 1965), 63-75.

40. Treynor, J. L., and K. K. Mazny (1966), "Can Mutual Funds Outguess the Market?" Harvard Business Review, (July-August, 1966) 131-136.

41. Tsiang, S. C. (1972), "The Rationale of the Mean Standard Deviation Analysis, Skewness Preference, and the Demand for Money," American Economic Review, 62, 354-71.